On Coloring the Square of Unit Disk Graph (DRAFT)

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ABSTRACT. A graph G is a Unit Disk (UD) graph if there is an assignment of unit disks centered at its vertices such that there is an edge in G iff the corresponding unit disks intersect. The square G^2 of G is defined as a graph on the same vertex set as G and having edges between pairs of vertices with graph distance at most two in G. Our results on the chromatic number of G^2 are motivated by an application for wireless sensor networks. Namely, if G, an UD, is an underlying graph of a wireless sensor network, the square of G, G^2, reflects possible conflicts in transmission between sensors in the network. An upper bound on the chromatic number of G^2 implies directly an upper bound on the frame length in the Time Division Multi-Allocation (TDMA) used for transmission scheduling in the sensor network.

The main result of the present paper shows that the chromatic number of G^2 is linear in $\omega(G)$ and is at most $13\omega(G)$, where $\omega(G)$ is the independence number of G. We also present an efficient algorithm to find a coloring corresponding to the upper bound.

1. Introduction and motivation

Unit Disk graph is the intersect graph of equal-sized disks in the plane[**4**]. It's the underlying graph of the Wireless Sensor Network (WSN) since the radio transmission range are modeled based on Euclidean distance, thus the radio coverage of a transmitter is modeled as a unit disk on the place. The connectivity between transmitters are determined by the intersection of the transmitters' radio coverage disk. Hence, in the modeling of WSN communication, each transmitter is mapped into a vertex; and there's a edge between u, v iff $d(u, v) \leq d$, where $d(u, v)$ denotes the Euclidean distance between u and v, d is the transmit range. Any two adjacent transmitters are considered interfering transmitters.

Graph coloring is important in wireless sensor network since in networks' initialization/setup phase, a time slot assignment would be given to all the transmitters in the network in the TDMA context to avoid radio conflicts. There are two type of collisions in a wireless sensor networks, one is called *direct collision*, in which any pair of neighbor transmitters being assigned same time slot, then this pair of transmitter would not hear anything from each other. The second type of collision is called *hidden collision*, in which transmitter wouldn't hear anything if two

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of its neighbors assigned same time slot, in other literatures, it is also called deconstructive radio interference. The underlying problem in terms of graph theory is a graph coloring problem, which is to color a graph such that any vertices within distance 2 would be colored differently. The problem is also called Radio Coloring Problem (RCP) since it is first originated in radio communication field, where transmitters those are close enough are assigned different radio frequency (instead of time slot in TDMA) to avoid radio interference.

Griggs and Yeh [**8**] introduced and formalized the problem as L(2,1)-labeling problem, such that in a radio connectivity graph any pair of vertices those have a distance of 2 or less would have to be assigned a different color. The L(2,1)-labeling problem need any vertices that are close enough, i.e. graph distance 1 apart to be colored at least two apart (here we number the colors with integer) and distance 2 apart to be colored at least 1 apart. The minimum number of colors is called λ. So, this problem is also called λ-coloring problem. In the wireless sensor network TDMA scheduling, we only need two vertices within distance 2 to be colored differently, thus, it is this equivalent to the L(1,1)-labeling problem. And, as we know, it is also equivalent to the coloring problem of square of a graph that has been stated in [**1**].

L-labeling problem has been very well studied in the past, as summarized by Bodlaender et. al in [**2**] bounds on variant graphs such as Grid, Tree, Bipartite, Split graph, Planar graph, Outplanar graphs all have been found. As for arbitrary graph, the best-known upper bound for λ-coloring is $\lambda < \Delta^2 + \Delta$ [**3**]. Other bounds on special graphs could also be found in [**2**] too.

Though, the above researchers looking for upper bound on arbitrary graph and variety of special graphs, we've found that the underlying graph of WSN, in general, does not fit into any special graph that is listed above, for example, it is too strong to assume the randomly deployed sensors would form a outerplanar or even a planar, neither does a tree. But, the bound on a arbitrary graph, which is $\Delta^2 + \Delta$, is relatively high for any practical use for our WSN scheduling and task planning. While we examine the property of the connectivity graph of geographically deployed Wireless Sensors Networks (WSN), we noticed since the connectivity are determined by their Euclidean distance, such that the underlying graph is only a subset of arbitrary graph. This type of graph, is the graph that has been presented in [**6**], which is called Unit Disk (UD) graph, and it could be generalized as Double Disk (DD) graph. The best know upper bound for chromatic number on DD graph is found 3ω in [**6**]. Other properties on the UD graphs are summarized in [**5**]. However, for UD graphs, to the best of our knowledge, solution on λ-coloring properties has not being presented, which is the goal of our paper.

As stated above, in this paper, motivation for this research is to find the bound on λ-coloring for WSN graph, which is for UD graphs. That would facilitate the research and industrial activities in TDMA scheduling for wireless sensor networks, so that it would lead to an optimal frame size, which is the key factor for maximize network throughput and least total response time.

The paper is organized as follows. In Section 2 we will define our problem and the auxiliary functions. In Section 3 we will show our cellular partition algorithm and prove of the Euclidean-Distance-Two coloring on UD graph. In Section 4 we will summarize and show the remarks of this work. At last, future work is stated in Section 5.

2. Auxiliary Definition

As shown in the above section, there have been intensive researches on the coloring square of trees and planar graphs, and outer-planar graphs, which lead to a linear solution. However, in WSN, sensor node could be deployed randomly, and, the radio radius could be set arbitrarily, in this case, a edges in a sensor network graph could be either as sparse as a barely connected graph, or as dense as a clique. So, none of the Grid, Tree, Planar would fit into our network model.
Another interesting observation is that for the researchers in the wireless communication field, since, the research in that field would always reference the result from graph theory, they have to put very strong assumptions to their model to achieve any practical result, like the paper on channel assignment in wireless communication [**9**], they have to assume the planarity of their wireless network. By observation, we know that our graph would not fit into any the specially graph that has been investigated in the L-labeling problem field. By the other hand, it is too expensive to have the $\Delta^2 + \Delta$ as our TDMA frame size if just use the general graph as our WSN model.
By simulation, we found that the coloring of this UD graphs result in a very low number of colors. Having seen this, we decide to investigate into this problem and find the reason.
So, first all of, we will construct a auxiliary problem to help with our G^2 problem.

DEFINITION 1. *Euclidean-Distance-Two graph (ED2)* of UD graphs. Euclidean-Distance-Two graph is a graph for UD graphs G, noted as G^{ED2}, such that there's an edge between any pair of vertices iff their Euclidean distance is less or equal to 2.

THEOREM 2.1. *For any UD graph G, the G^2, known as square of graph G, there's $G^2 \subseteq G^{ED2}$.*

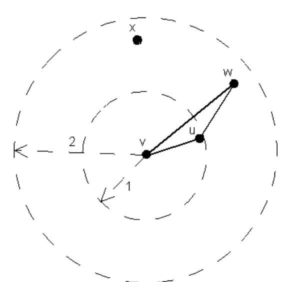

FIGURE 1. Square of graph G

PROOF. For any vertex v in UD graph G, $\forall w$, such that $d(v, w) = 2$, (where $d(v, w)$ denotes the distance between vertex v and w). As shown in figure 1, there must $\exists u \in G$, such that both $Edge(v, u)$ and $Edge(u, w) \in G$, and since G is a UD graph, there's $d_{ED}(v, w) \leq d_{ED}(v, u) + d_{ED}(u, w) \leq 2$, (where $d_{ED}(v, u)$ denotes the Euclidean distance between v and u). Then $Edge(v, w) \in G^{ED2}$. Therefore G^2, is a sub-graph of G^{ED2}.
Inequality could happen as shown in figure 2, there might $\exists x \in G^{ED2}$, such that $1 < d_{ED}(v, x) \leq 2$, and there does not $\exists u$, such that $Edge(v, u) \in G$ and $Edge(u, x) \in G$. $\square$

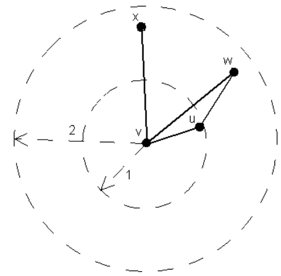

FIGURE 2. Euclidean-Distance-Two graph of G

COROLLARY 2.2. *For any UD graph G, a coloring scheme $\chi_{ED2}(G)$ for coloring G^{ED2} would also color G^2, which is equivalent to $L(1,1)$-labeling of G.*

PROOF. Since we've proved in theorem 2.1, any G^2 is a sub-graph of G^{ED2}, then, there exist a coloring scheme $\chi_{ED2}(G)$ fulfill G^{ED2} would be sufficient for G^2. Since, $L(1,1)$-labeling of G is equivalent to the coloring G^2, we have $\chi_{ED2}(G)$ fulfill $L(1,1)$-labeling of G. $\square$

3. Cellular Partition Algorithm

The penalty of the bound on G^2 on general graph is actually brought by the Δ degree on the one hop of any node, it is true that in an arbitrary graph, there could be no overlap between two adjacent node of their adjacent like bipartite, however, in a UD graph, considerable big number of neighbor is in a shared manner, so that the total number of the two-hop neighborhood could be much less. By seeing this, we constructed a partition algorithm, such that we could effectively bound the two-hop neighborhood in a way that, they would not be enormously big. In the above section, by defining the G^{ED2} graph from the original graph, we showed that the two-hop neighborhood is bounded by the Euclidean distance 2, such that the highest degree of a node in the G^{ED2} would not exceed the total number of node we could place into its radius 2 circle. And that, is a easy bound for the chromatic number on G^{ED2} by [**10**]

DEFINITION 2. *Cellular Partition Algorithm*

By doing the cellular partition the plane, we create "buckets" as the size of a unit hexagon with a side length of 1/2, and thus, the diagonal length is 1. In this bucket, any vertices inside will form a clique, since there are no any two vertices within the same hexagon has a Euclidean distance of more than 1. Knowing the maximum clique size is ω, we can determine that for any vertices' deployment, there would not be more than ω vertices in the same bucket.
While successfully partition a UD graph into non-overlap buckets, we could find a way to color those buckets, since we know that ω colors is sufficient to color each bucket.

THEOREM 3.1. *Cellular Partition Algorithm could color Euclidean-Distance-Two graph G^{ED2} for any UD graph G in 13ω.*

PROOF. We choose hexagon to cover the whole plane based on the observation and prove that Hexagon is the biggest polygon that could cover the plane only with its own. Therefore, it is the biggest tile that could cover the plane with no overlap, so that it is the most efficient way to cover the plane. As described in the above section, we partition the whole plane into hexagons with diameter of 1. All vertices being partitioned in that hexagon would form a clique. Since, ω is the max clique size, so, we could place at most ω into each bucket.

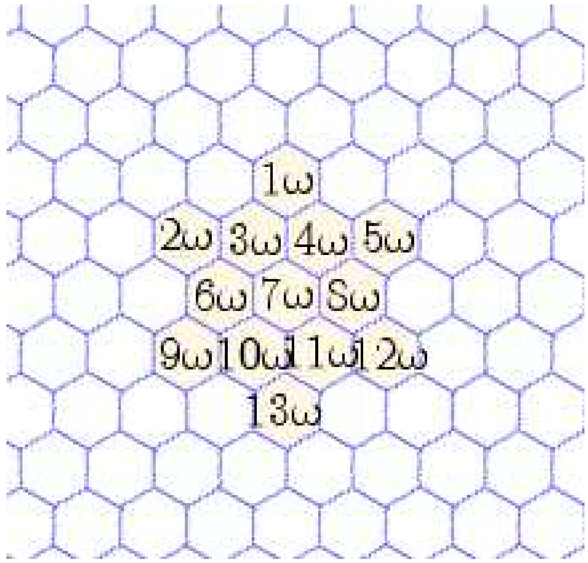

FIGURE 3. A patch with 13 hexagons

Next, we will construct a patch with 13 hexagons, and assign 13ω colors to the patch, an example of the patch is shown in figure 3 and keep the same orientation, and use the patch to pave the whole plane as shown in figure 4. In the next section, We would prove that, hexagon with number ith in a patch would be at a Euclidean distance of at least 2 to the same number ith hexagon in adjacent patches.

Observed that we keep patches orientation while we use them to pave the whole plane, for any hexagon of same number in the different patches, the distance is a constant. For simplicity, We will compute the center hexagon's shift distance, and other hexagon's shift is the same. Left side of figure 5 shows the distance between the adjacent patches. Right side of figure 5, A and B are the center hexagon from the adjacent patches, their distance is computed as:

$AB^2 = BC^2 + AC^2$

since $AC = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

, and $BC = \frac{7}{2} * \frac{\sqrt{3}}{2}$

then $AB = \frac{\sqrt{156}}{4} \approx 3.12 > 3$.

Since the vertices in both center hexagon A and B could only be away from $\frac{1}{2}$ to their center, then, the distance from any vertex in center hexagon A to any vertex in center hexagon B would be greater than 2. Therefore, our patch deployment fulfills the coloring of G^{ED2}.

Therefore, by applying the above coloring scheme, we could prove that by using these 13 hexagon patches, we could successfully cover the whole plane, and since that each hexagon in a certain color is far away from its sibling hexagon with same

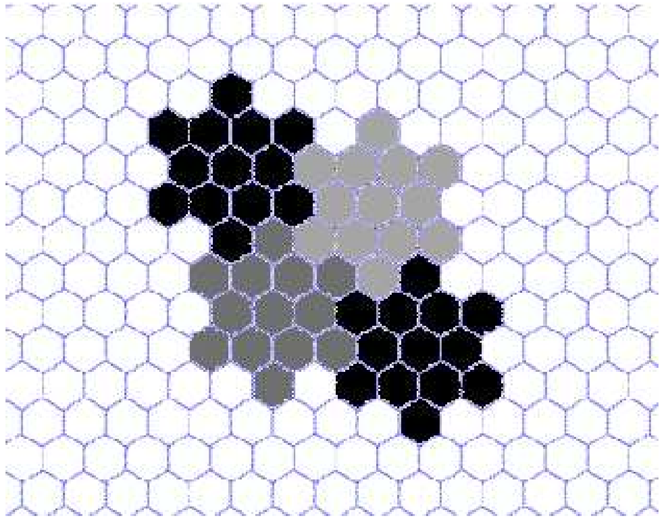

FIGURE 4. A patch with 13 hexagons that covers the plane.

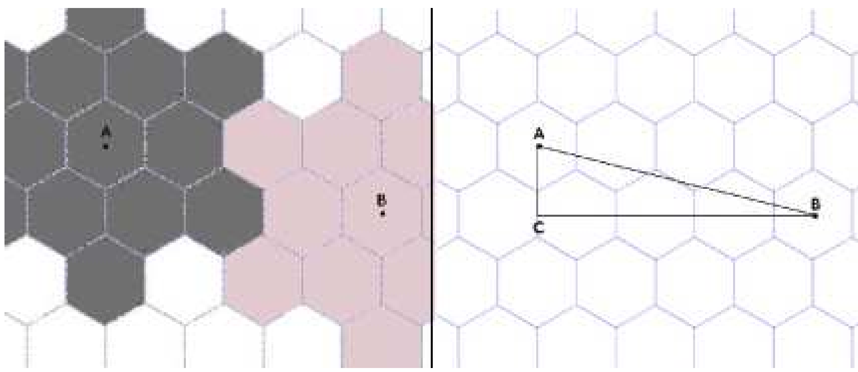

FIGURE 5. The center distance of different patches

color, it is a valid Euclidean-Distance-Two coloring. Use the Theorem 2.1, it is also a valid coloring scheme for the square of the original graph, G^2. □

3.1. A lower-upper bound on Theorem 3.1. In the above section, we showed a upper bound of the G^2, however, in our simulation, this upper bound is relatively high to achieve, in the following section, we would construct a example that shows how high in practice the upper bound could reach. A circular example We could construct a vertex deployment, such that there are 20 vertices distributed evenly on a circle with diameter of $\sqrt{2}$. In this case, the ω=6, Δ=10, and it needs 20 colors for G^2. It is trivial to show that in general we could always construct such a circular deployment with even more vertices, such that $\Delta = 2\omega - 2$, $L(1,1)$-labeling of $G = 4\omega - 4$, so that we have a 4-approximation on coloring G^2.

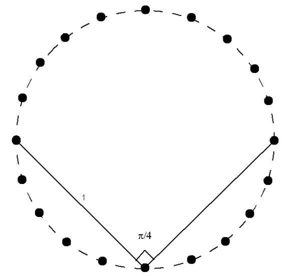

FIGURE 6. Circular example for a lower-upper bound on G^2.

4. Conclusions and Remarks

Comments on the DD graph work by [**6**]. They come up with a DD solution, which is very similar to our Euclidean distance two graph, and we could even mimic the situation by fixing the out-circle with radius 1.5, and the inner-circle with radius 0.5. However, in their problem setup, they did not differentiate the coloring scheme with the respect of maximum clique size, such that the solution we have could not directly be derived from their result. In this paper, we considered the coloring problem on square of UD graphs, which extended the result from [**7**]. We have shown an upper bound of 13ω for this problem.

5. Future work

A major open question is whether the 13ω could be improved and how. We realize that the solution we presented in this paper has not fully utilize the distance two property from the original problem, we constructed a sufficient graph G^{ED2} to solve the problem. However, since it is not necessary, and also, by our simulation, we've found that the practical total number of colors we ended up with is far lower than the bound that we've derived in the above section. We hope there could be a better way to properly apply the distance two property and derive a tight bound for the UD graphs.

Another future direction would be to unleash the general λ-coloring solutions, such as $L(2,1)$-labeling on UD graphs, since, in practical use, there's issue like timer drifting happened to the low-cost device on the sensors, a difference of two on time slot assignment between adjacent sensor nodes would make the network more robust to the time drift on the nodes. It would be very useful if we could come up with a solution on L(2,1)-labeling problem with relatively small additional cost to L(1,1)-labeling on UD graphs.

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